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ON THE GLOBAL STABILITY OF THE LORENTZ SYSTEM*

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Conditions for global asymptotic stability are obtained for the set of Lorentz equations that generalize the theorem of V.I. Iudovich.

Consider the set of equations

$$\sigma^{\cdot} = \eta, \quad \eta^{\cdot} = -g(\eta, \sigma) + z^* Cf(\sigma) - \varphi(\sigma), \quad z^{\cdot} = Az + Bf(\sigma)\eta \tag{1}$$

where A is a constant $(n \times n)$ matrix whose eigenvalues have negative real parts, B and C are constant $(n \times m)$ matrices, $f(\sigma)$ is a continuously differentiable m-dimensional vector function, and $\varphi(\sigma)$ and $g(\eta, \sigma)$ are continuously differentiable functions. The set of equations (1) defines the operation of a synchronous generator and the Bouass-Sadre governor /1,2/. Using the transformations of phase variables close to the Iudovich transormation /3/, the Lorentz set of equations may also be written in the form (1).

Henceforth, we assume that any solution of (1) is defined in the interval $(0, +\infty)$, its stationary set Λ consists of isolated points, and

$$g(\eta, \sigma)\eta \geqslant \mu_1\eta^2, \quad \forall\eta \in R^1, \quad \forall\sigma \in R^1$$

where μ_1 is some positive number.

We introduce into the analysis the matrix $K(p) = C^*(A - pI)^{-1}B$, where p is a complex number, I is a unit matrix, and μ is some positive number.

Theorem. Let the inequality

$$\det[\mu_1\mu^{-1}I + \operatorname{Re} K(i\omega)] \neq 0 \tag{2}$$

hold for all $\omega \in R^1$. Then any solution $x(t) = \| \sigma(t), \eta(t), z(t) \|$ of system (1) bounded in $(0, +\infty)$ satisfying condition $\overline{\lim}_{t\to+\infty} |f(\sigma(t))|^2 < \mu$, approaches some state of equilibrium as $t \to +\infty$.

Proof. The inequality (2) is equivalent to the relation

$$\mu_1\mu^{-1}I + \operatorname{Re} K(i\omega) > 0, \quad \forall\omega \in R^1$$

According to Theorem 1.2.7 and Lemma 1.2.3 in /2/ from this it follows that an $n \times n$-matrix $H = H^* > 0$ and a number $\varepsilon > 0$ exist which satisfy the inequality

$$2z^*H(Az + B\xi) + z^*C\xi - \mu_1\mu^{-1}|\xi|^2 \leqslant -\varepsilon|z|^2, \quad \forall z \in R^n, \quad \forall\xi \in R^m \tag{3}$$

Consider the function

$$V(x) = z^*Hz + \frac{1}{2}\eta^2 + \int_0^\sigma \varphi(\sigma)\,d\sigma$$

The inequality (3) implies that for the solutions $x(t)$ of (1) we have the estimate

$$V^{\cdot}(x(t)) \leqslant -\mu_1\eta(t)^2(1 - \mu^{-1}|f(\sigma(t))|^2) - \varepsilon|z(t)|^2 \tag{4}$$

from which it follows that for the bounded trajectory $x(t, x_0)$ of (1) that satisfies the condition

$$\overline{\lim}_{t\to+\infty} |f(\sigma(t, x_0))|^2 < \mu \tag{5}$$

(here $x(0, x_0) = x_0$) the function $V(x(t, x_0))$ does not increase with respect to t in some interval $(\tau, +\infty)$. From this and from the boundedness of $V(x(t, x_0))$ it follows that a finite limit $\lim_{t\to+\infty} V(x(t, x_0)) = L$ exists.

The trajectory $x(t, x_0)$ is bounded in $(0, +\infty)$, hence the set Ω of its limit points is nonempty. Let $y \in \Omega$. From /2/ we know that the trajectory $x(t, y) \in \Omega$, $\forall t \in R^1$. Hence $V(x(t, y)) = L$, $\forall t \in R^1$. Moreover, from condition (5) it follows that $|f(\sigma(t, y))|^2 < \mu$, $\forall t \in R^1$. But then, using (4) we obtain the indentity $z(t, y) \equiv 0$, $\eta(t, y) \equiv 0$ From (1) and $\eta(t, y) \equiv 0$ we also have $\sigma(t, y) \equiv \mathrm{const}$. Thus $\Omega \subset \Lambda$. This inclusion and the isolation of the points Λ prove the theorem.

Let us now consider the Lorentz set of equations

$$\frac{dx_1}{dt_1} = -\sigma_1(x_1 - y_1), \quad \frac{dy_1}{dt_1} = -x_1z_1 + r_1x_1 - y_1, \quad \frac{dz_1}{dt_1} = x_1y_1 - b_1z_1 \tag{6}$$

where σ_1, r_1, b_1 are positive numbers and $r_1 > 1$. Using a substitution similar to that of

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Iudovich /3/.

$$\sigma = \frac{\varepsilon}{\sqrt{2\sigma_1}} x_1, \quad \eta = \frac{\varepsilon^2}{\sqrt{2}}(y_1 - x_1), \quad z = \varepsilon^3\left(z_1 - \frac{x_1^2}{b_1}\right), \quad t = \frac{\sqrt{\sigma_1}}{\varepsilon} t_1$$
$$\mu_1 = \varepsilon\frac{(\sigma_1+1)}{\sqrt{\sigma_1}}, \quad A = -\varepsilon\frac{b_1}{\sqrt{\sigma_1}}, \quad \varepsilon = (r_1 - 1)^{-1/2}, \quad \beta = \varepsilon\frac{(2\sigma_1 - b_1)}{\sqrt{\sigma_1}}$$

we reduce Eqs.(6) to Eqs.(1) with $B = 2\beta A^{-1}$, $C = -1$, $g(\eta, \sigma) = \mu_1\eta$, $f(\sigma) = \sigma$, $\varphi(\sigma) = (1 - \beta A^{-1})\sigma^3 - \sigma$. Here $K(p) = 2\beta(p - A)^{-1}A^{-1}$. Hence inequality (2) is satisfied, if $\beta < \mu_1 A^2 (2\mu)^{-1}$. This estimate can be written as follows:

$$2\sigma_1 - b_1 < \frac{(\sigma_1 + 1) b_1^2}{2\sigma_1(r_1 - 1)}\mu^{-1} \tag{7}$$

Since Eqs.(6) are dissipative /3/, from (7) and the theorem we have the following corollary:

Corollary. 1. If the solution $x_1(t), y_1(t), z_1(t)$ of Eqs.(6) satisfies the condition

$$(2\sigma_1 - b_1)\overline{\lim}_{t\to+\infty} x_1(t)^2 < b_1^2(\sigma_1 + 1) \tag{8}$$

then, as $t \to +\infty$, it approaches some equilibrium position.

We introduce the notation

$$\gamma = \sigma_1 r_1^{-1}, \quad \lambda_0 = \min(1, b_1, \sigma_1), \quad \lambda \in [0, \lambda_0]$$
$$\theta = 2[\sigma_1 - \sqrt{\gamma(\sigma_1 - \lambda)(1 - \lambda)}],$$
$$W(x_1, y_1, z_1) = \frac{1}{2}[x_1^2 + \gamma(y_1^2 + z_1^2)] - \theta z_1$$
$$\Gamma = \frac{\theta^2(b_1 - 2\lambda)^2}{8\lambda\gamma(b_1 - \lambda)}, \quad \alpha = \frac{b_1^2[\sqrt{r_1\sigma_1} - \sqrt{(\sigma_1 - \lambda)(1 - \lambda)}]^2}{\lambda(b_1 - \lambda)(1 + \gamma)}$$

(A function W of similar form was used to prove that the Lorentz system is dissipative in /3-5/.)

Lemma 1. For any solution $x_1(t), y_1(t), z_1(t)$ of Eqs.(6) the inequality

$$\overline{\lim}_{t\to+\infty} W(x_1(t), y_1(t), z_1(t)) \leqslant \Gamma \tag{9}$$

holds.

Proof. It is evident that

$$W^{\cdot} + 2\lambda W = -(\sigma_1 - \lambda)x_1^2 - \gamma(1 - \lambda)y_1^2 - \gamma(b_1 - \lambda)z_1^2 + [\sigma_1 + \gamma r_1 - \theta]x_1y_1 + \theta(b_1 - 2\lambda)z_1 \leqslant -\gamma(b_1 - \lambda)z_1^2 + \theta(b_1 - 2\lambda)z_1 \leqslant 2\lambda\Gamma$$

Hence $(W - \Gamma)^{\cdot} + 2\lambda(W - \Gamma) \leqslant 0$ and, consequently,

$$W(x_1(t), y_1(t), z_1(t)) - \Gamma \leqslant [W(x_1(0), y_1(0), z_1(0)) - \Gamma]e^{-2\lambda t}, \quad \forall t \geqslant 0$$

which means that inequality (9) is satisfied.

We will introduce into the investigation the sets

$$\Phi_1 = \{x_1, y_1, z_1 \mid W(x_1, y_1, z_1) \leqslant \Gamma, x_1 > \sqrt{\alpha}\}$$
$$\Phi_2 = \{x_1, y_1, z_1 \mid W(x_1, y_1, z_1) \leqslant \Gamma, x_1 < -\sqrt{\alpha}\}$$

Lemma 2. If at the point t the solution $x_1(t), y_1(t), z_1(t)$ of (6) is contained in Φ_1, then $x_1^{\cdot}(t) < 0$, and if it is contained in Φ_2, then $x_1^{\cdot}(t) > 0$.

Proof. Considering the first part of the lemma and assuming the opposite, from the equation $x_1^{\cdot} = -\sigma_1(x_1 - y_1)$ we obtain that $y_1(t) \geqslant x_1(t) > \sqrt{\alpha}$. Hence

$$W(x_1(t), \quad y_1(t), \quad z_1(t)) > \frac{1}{2}\left[\alpha + \gamma\alpha + \gamma\left(z_1(t) - \frac{\theta}{\gamma}\right)^2 - \frac{\theta^2}{\gamma}\right] \geqslant \frac{1}{2}\left[\alpha(1 + \gamma) - \frac{\theta^2}{\gamma}\right] = \Gamma$$

which contradicts the statement that the solution considered at the point t is contained in Φ_1.

The second part of the lemma is proved similarly.

Lemmas 1 and 2 imply that for any solution $x_1(t), y_1(t), z_1(t)$ of (6) the following estimate holds:

$$\overline{\lim}_{t\to+\infty} x_1(t)^2 \leqslant \frac{b_1^2}{(1 + \sigma_1 r_1^{-1})}\min_{\lambda\in[0,\lambda_0]}\frac{[\sqrt{r_1\sigma_1} - \sqrt{(\sigma_1 - \lambda)(1 - \lambda)}]^2}{\lambda(b_1 - \lambda)} \tag{10}$$

Comparing (8) and (10) we obtain the following statements.

Corollary 2. If

$$2\sigma_1 - b_1 < (\sigma_1 + 1)(\sigma_1 r_1^{-1} + 1)\max_{\lambda\in[0,\lambda_0]}\frac{\lambda(b_1 - \lambda)}{[\sqrt{r_1\sigma_1} - \sqrt{(\sigma_1 - \lambda)(1 - \lambda)}]^2} \tag{11}$$

then (6) is over-all asymptotically stable, i.e. any of its solutions approach some stable equilibrium as $t \to +\infty$.

The estimate (11) is a generalization of a similar estimate by Iudovich /3/: $2\sigma_1 - b_1 < 0$ which was obtained using the Liapunov function.

Note 1. When $r_1 \to 1$, the right-hand side of inequality (11) approaches $+\infty$. Hence for fixed b_1 and σ_1 an $r_1 > 1$, will always be found that satisfies condition (11).

Note 2. Sometimes it is interesting to consider small b_1 /4/. Selecting in that case $\lambda = b_1/2$ we obtain from (11) the following condition of global asymptotic stability:

$$r_1 < 1 + \frac{(\sigma_1 + 1)}{\sqrt{2\sigma_1}} b_1$$

REFERENCES

1. LEONOV G.A., On one class of dynamic systems with cylindrical phase space. Sib. Matem. Zh. Vol. 15, No. 1, 1976.
2. GELIG A. Kh., LEONOV G.A. and IAKUBOVICH V.A., Stability of Non-linear Systems with a Non-unique State of Equilibrium. Moscow, NAUKA, 1978.
3. BELYKH V.N., Qualitative Methods Of the Theory of Non-linear Oscillations of Lumped Systems. Uchebn. Posobie. Gor'kii, Izd. Gor'k. Univ., 1980.
4. Strange attractors. Collect. of Articles. Moscow, MIR, 1981.
5. RUELLE D., The Lorenz attractor and the problem of turbulence. In: Lecture Notes in Math., No. 565, Springer, 1976.

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THE LAWS OF VARIATION OF ENERGY AND MOMENTUM FOR ONE-DIMENSIONAL SYSTEMS WITH MOVING MOUNTINGS AND LOADS*

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The self-consistent dynamic behaviour of a one-dimensional system with a moving load is considered. The natural boundary conditions, previously obtained from the variational Hamiltonian principle /1/ for the self-consistent problem of the dynamics of one-dimensional systems, when the boundary motions are not specified, are used to show that the motion of the load results in the appearance of additional forces that are proportional to the load velocity. Expressions are obtained in terms of the density of the Lagrange function for the wave pressure, the wave energy flux, the wave momentum, the energy transport velocity, the work of the forces that vary the system parameters, and the distributed recoil forces that occur when waves propagate along a non-uniform system. The radiation conditions are discussed in the class of systems considered. The critical velocities of the load moving along a Timoshenko beam are determined.

1. Consider a holonomic system with ideal constraints, consisting of a one-dimensional system along which a concentrated load is displaced, consistent with the motion of a distributed system.

Let x be a Cartesian coordinate along the one-dimensional system, t be the time, and $D = \{(x, t): a \leqslant x \leqslant b,\ \alpha \leqslant t \leqslant \beta\}$ be some rectangular region in the plane x, t. We assume that the motion of the load is defined by some function $x = \chi(t)$, doubly differentiable in $[\alpha, \beta]$, such that the curve $K = \{(x, t): x = \chi(t), \alpha \leqslant t \leqslant \beta\}$ lies in region D and divides it into parts as follows:

$$D_1 = \{(x, t):\ a \leqslant x \leqslant \chi(t),\quad \alpha \leqslant t \leqslant \beta\},\qquad D_2 = \{(x, t):\ \chi(t) \leqslant x \leqslant b,\ \alpha \leqslant t \leqslant \beta\}$$

and that the law of motion of the distributed system is defined by some vector function continuous in D

$$\mathbf{u}(x,t) = \begin{cases} \mathbf{u}^1(x, t) = (u_1^1(x, t), \ldots, u_n^1(x, t)), & (x, t) \in D_1 \\ \mathbf{u}^2(x, t) = (u_1^2(x, t), \ldots, u_n^2(x, t)), & (x, t) \in D_2 \end{cases}$$

where the vector functions $\mathbf{u}^1(x, t)$ and $\mathbf{u}^2(x, t)$ are doubly continuously differentiable in regions D_1 and D_2, respectively.

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